ANALYTICAL EXPRESSIONS FOR AVERAGE BIT STATISTICS OF SIGNAL LINES IN DSP ARCHITECTURES*

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Abstract

Accurate high-level power estimation methods are required for exploring the design space to obtain an optimal low-power circuit. DSP architectures are regular and they consist of interconnected macro-blocks such as adders and multipliers. In [1], the power dissipation of macro-blocks was related to the average bit statistics. Given the input word-level statistics for a DSP architecture, the word-level statistics at all the internal signal lines can be computed analytically using transfer function evaluation or by propagating the statistics. In this paper, we present simple analytical expressions for computing the average bit statistics using the word-level statistics of the signal lines in a DSP architecture.

1 Introduction

The decreasing feature size and other developments in device technology have led to the design of complex circuits operating at high clock rates. These circuits have a higher power dissipation per unit area. High power densities lead to higher operating temperatures which decrease the lifetime of IC due to reliability problems. For reliable long-term operation, sophisticated and expensive packaging is required to ensure proper heat dissipation. The integrated circuits used in portable products, such as laptop computers and cellular phones need to consume less power in order to extend the battery life and reduce the overall weight of the final product. Hence, power minimization is an important concern in present day IC design.

An accurate estimate of average power dissipation at a high-level (RTL level or architectural level) is necessary for the exploration of the design space to obtain a power optimal design. DSP architectures are modular, regular and they consist of instantiations of DSP macro-blocks such as adders, multipliers, multiplexers. The word-level statistics of all internal signals can be computed from the input signal statistics using transfer function evaluation or propagating the input statistics. The power dissipation of a macro-block can be shown to be a function of the average bit statistics of the output and the input signal lines [1]. The total transition activity and the average bit transition activity of the bits can be computed by using the methods proposed in [2,3]. Both these methods do not present closed-form analytical expressions for bit statistics. In this work, we *derive* analytical expressions for bit statistics and develop simple and accurate closed-form analytical expressions for average bit statistics in terms of the word-level statistics.

Signals in DSP systems are represented in finite precision using a certain signal encoding. There exist a number of different representations such as the sign magnitude, one's-complement and two's-complement, but the two's-complement representation is the most commonly used representation [4]. The word values of a signal have a certain probability distribution. The bit statistics depend on both the signal probability density function and the type of encoding. Many DSP inputs can be closely approximated by a Gaussian process. In this work, we assume that the input is a zero-mean Gaussian signal with a two's-complement number representation. Many signals in real-life applications are zero-mean Gaussian signals. For audio signals, such as speech or music, the distribution of the amplitude is concentrated about zero and falls off rapidly with increasing amplitude [4]. Hence, these signals can be considered as zero-mean Gaussian signals. Given two signals with the same variance, for the same quantization error the signal with non-zero mean would require more bits than the signal with zero mean. Signals with a non-zero mean do not utilize the entire dynamic range. Even if the input signal is not a zero-mean signal, the mean value can be subtracted from the input and an appropriately scaled value of the mean can be added to the output of the DSP circuit.

If the input signal is a zero-mean Gaussian signal, then all the internal signals in a DSP architecture are also zero-mean Gaussian signals. Since, we consider zero-mean signals, the word-level statistics of a signal consist of the variance (σ^2) and the correlation coefficient (ρ). The bit statistics are the transition activity (t_i), probability (p_i) and the temporal correlation (ρ_i) respectively, where t_i denotes the probability of a transition of the bit, p_i denotes the probability that the bit is 1 (logic HIGH), and ρ_i denotes the lag-one temporal correlation of the bit. In this work, we present analytical expressions for the average bit statistics, *average bit probability*, *average bit temporal correlation* and *average bit transition activity* in terms of the signal word-level statistics (σ, ρ).

This paper is organized as follows. In the next section, we present expressions for the bit probability and the average bit probability. In section 3, we give the analytical expression for average bit transition activity. In section 4, we derive a relationship between the average bit transition activity and average bit temporal correlation. Finally, in section 5 we give the conclusions.

2 Average Bit Probability

In this section, we first present a theorem which relates the average bit probability of signals to certain properties of the signal. We then show that the theorem is valid for a zero-mean Gaussian signal in two's complement representation.

Consider a signal represented using a word-length of B bits. The signal values are represented using 2^B distinct binary representations. Let $N = 2^{B-1}$ and $\{a_1, a_2, \cdots, a_N\}$, $\{b_1, b_2, \cdots, b_N\}$ denote two sets of N symbols, where each of the symbols corresponds to a particular bit representation.

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Let $P(a_i), P(b_i)$ denote the probability values of the symbols a_i, b_i, respectively. The following theorem relates the bit probability to certain properties of the symbols $a_1, a_2, \cdots, a_N$, $b_1, b_2, \cdots, b_N$.

Theorem 1 *If the symbols $a_1, a_2, \cdots, a_N$, $b_1, b_2, \cdots, b_N$ of a signal are such that, for each a_i there is a b_i with $P(a_i) = P(b_i)$ and the bit representations of a_i, b_i are complements of each other then, bit probability is 0.5 for each of the B bits.*

Proof: To prove the theorem, we use the conditions that the symbols a_1, a_2, $\cdots$, a_N, b_1, b_2, $\cdots$,b_N satisfy and the fact that the cumulative sum of the probability values associated with the symbols a_i's and b_i's is 1, *i.e.*,

$$\sum_{i=1}^{N}\{P(a_i) + P(b_i)\} = 1. \tag{1}$$

We will prove the above theorem for an arbitrary bit j. The bit j is 1 in the bit representation of exactly N symbols and 0 for the bit representation of the other N symbols. This is due to the fact that we have 2^B (2N) distinct binary representations for the signal values. Let S_j denote the set of symbols for which the bit j is 1 and $\overline{S_j}$ denote the set of symbols for which the bit j is 0. Since the symbols have distinct bit representations the sets $S_j, \overline{S_j}$ are disjoint sets. The probability of a bit j, denoted by p_j is given by the following expression,

$$p_j = \sum_{a_i \in S_j} P(a_i) + \sum_{b_i \in S_j} P(b_i). \tag{2}$$

The set S_j cannot contain symbols that have bit representations that are complements of each other. It would violate the condition that S_j contains symbols for which the bit j is 1. Hence for every symbol $a_i \in S_j$, there is a symbol $b_i \in \overline{S_j}$ that has the complement bit representation of symbol a_i. Since, the probability values of the symbols a_i and b_i are the same, we can relate the sum of the probability values of the symbols $a_i \in S_j$ and $b_i \in \overline{S_j}$ by the following equation,

$$\sum_{a_i \in S_j} P(a_i) = \sum_{b_i \in \overline{S_j}} P(b_i). \tag{3}$$

Similarly, the sum of the probability values of the symbols $b_i \in S_j$ and $a_i \in \overline{S_j}$ are related by the following equation,

$$\sum_{b_i \in S_j} P(b_i) = \sum_{a_i \in \overline{S_j}} P(a_i). \tag{4}$$

Since S_j and $\overline{S_j}$ are disjoint sets that contain all the symbols, Eqn. (1) which is an expression for the cumulative sum of the probability values associated with all the symbols can be rewritten as,

$$\sum_{a_i \in S_j} P(a_i) + \sum_{b_i \in S_j} P(b_i) + \sum_{a_i \in \overline{S_j}} P(a_i) + \sum_{b_i \in \overline{S_j}} P(b_i) = 1 \tag{5}$$

Using (3) and (4) in (5),

$$2\sum_{a_i \in S_j} P(a_i) + 2\sum_{b_i \in S_j} P(b_i) = 1 \tag{6}$$

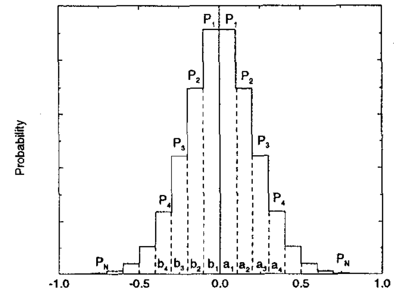

Figure 1: Probability distribution of the word values

Using (2) and (6), the expression for bit probability (p_j) can be written as,

$$p_j = \sum_{a_i \in S_j} P(a_i) + \sum_{b_i \in S_j} P(b_i) = 0.5 \tag{7}$$

■

Theorem 1 relates the probability of the bits to the properties of the symbols which correspond to a binary representation of the signal word value. The properties of the symbols depend on the probability density function and the type of encoding. In general, a signal with any probability density function that is symmetric about zero amplitude and uses either two's-complement or one's-complement signal encoding with sufficient number of bits satisfies the conditions on the symbols given in *Theorem 1.* Hence, by *Theorem 1* the probability of each of the bits is 0.5 for all such signals. We will show this result for the two's-complement representation of a zero-mean Gaussian signal.

Corollary 1 *For a zero-mean Gaussian signal, the bit probability (p_i) in the two's-complement representation is* 0.5

Proof: Let $a_1, a_2, \cdots, a_N$ denote the two's-complement representation of the positive signal word values in the increasing order. Let $b_1, b_2, \cdots, b_N$ denote the two's-complement representation of the negative signal word values in the decreasing order. Figure 1 shows the notations described. Note that the bit representations of (a_i, b_i) $\forall i$, are complements of each other. Let $P_1, P_2, \cdots, P_N$ denote the probability values $P(a_i)$ corresponding to each of the a_i's. A Gaussian signal is symmetric about the mean and hence, we can relate $P(a_i)$ and $P(b_i)$. This relationship is shown below.

$$P(a_i) = P(b_i) = P_i \ \ \forall i \tag{8}$$

Since the conditions on the symbols specified in theorem 1 are satisfied, the probability of each of the bits is 0.5.

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By *Corollary 1*, the probability of each of the bits in the two's-complement representation of a zero-mean Gaussian signal is 0.5. Hence the average value of the bit probability is also 0.5.

3 Average Bit Transition Activity

In this section, we first present a theorem that relates the probability of a sign change to the word-statistics of the signal. We then obtain the expression for the most significant bit (MSB) transition activity. The MSB bit transition activity and the break-points presented in [2] are used to obtain the expression for average bit transition activity.

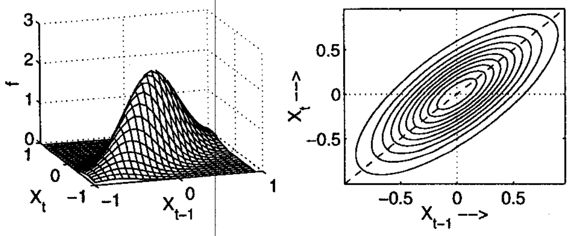

Figure 2: Probability density function and contour plot of jointly Gaussian random variables (X_t, X_{t-1})

The average bit transition activity is related to the word-statistics of the signal. Let $P(t^{-\to +})$ denote probability of a sign change from a negative signal value to a positive signal value. Let $P(t^{+\to -})$ denote probability of a sign change from a positive signal value to a negative signal value. The value of the sign change probability is dependent on the word-statistics. The following theorem relates the sign change probability and the word-statistics.

Theorem 2 *Given random variables (X_t, X_{t-1}) that are jointly Gaussian with zero-mean, variance σ^2 and correlation coefficient ρ, the probability of sign change is given by,*

$$P(t^{-\to +}) = P(X_{t-1} < 0, X_t \geq 0) = \frac{1}{2\pi} cos^{-1}(\rho) \quad (9)$$

$$P(t^{+\to -}) = P(X_{t-1} \geq 0, X_t < 0) = \frac{1}{2\pi} cos^{-1}(\rho) \quad (10)$$

Proof: The joint density function f of the random variables (X_t, X_{t-1}) is given by,

$$f(X_t, X_{t-1}) = \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} \; e^{\left(-\frac{X_t^2 + X_{t-1}^2 - 2\rho X_t X_{t-1}}{2\sigma^2(1-\rho^2)}\right)} \quad (11)$$

Fig. 2 shows the plot of the joint density function and the contour plot. Since the joint probability density function is symmetric, $P(t^{+\to -})$ is equal to $P(t^{-\to +})$. Hence, we will prove the probability of sign change from a negative to positive signal value ($P(t^{-\to +})$). The probability of sign change $P(t^{-\to +})$ is given by,

$$P(t^{-\to +}) = P(X_{t-1} < 0, X_t \geq 0) \quad (12)$$

This can be obtained by finding the volume under the surface in the region $(X_{t-1} < 0, X_t \geq 0)$ and dividing by the total volume under the surface. In this proof, we consider a particular contour and find the ratio of the area in the desired region to the total area of the contour. Furthermore, we show that this ratio is independent of the contour chosen. Hence, this ratio is valid for all contours and it also denotes the ratio of the volume in the desired region to the total volume. The equation for a contour is given by,

$$X_t^2 + X_{t-1}^2 - 2\rho X_t X_{t-1} = k^2 \quad (13)$$

where k is a constant. The ratio of the shaded area to the total area in Fig. 3 gives the probability of sign change for this contour. Put $Z_1 = (X_t + X_{t-1}/(\sqrt{2}k)$ and $Z_2 = (X_t - X_{t-1}/(\sqrt{2}k)$. Using (Z_1, Z_2) and performing the change of variable, Eqn. (13) becomes,

$$(1-\rho)Z_1^2 + (1+\rho)Z_2^2 = 1 \quad (14)$$

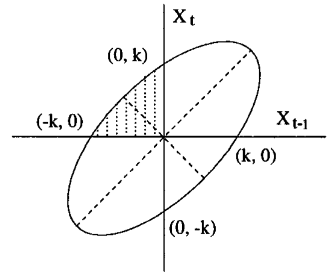

Figure 3: Contour plot for a particular value of k

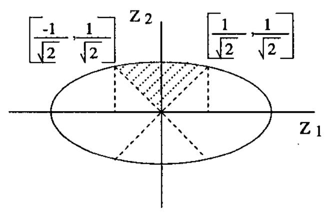

Figure 4: Transformed contour plot

Fig. 4 shows the plot of the above equation and the corresponding end points. Eqn. (14) can be rewritten as,

$$\frac{Z_1^2}{a^2} + \frac{Z_2^2}{b^2} = 1 \quad (15)$$

a, b denote $1/\sqrt{1-\rho}$ and $1/\sqrt{1+\rho}$ respectively. Let A_s denote the area of the shaded region. The value of A_s can be obtained by integrating the area under the curve $Z_2 = \frac{b}{a}\sqrt{a^2 - Z_1^2}$, and deleting the area of the two right triangles. The following expression relates the area under the curve and the area under the shaded region denoted by A_s.

$$A_s = -\frac{1}{2} + \frac{b}{a}\int_{-1/\sqrt{2}}^{1/\sqrt{2}} \sqrt{a^2 - Z_1^2} \; dZ_1 \quad (16)$$

Solving the integral yields,

$$A_s = \frac{cos^{-1}(\rho)}{2\sqrt{1-\rho^2}} \quad (17)$$

The total area of the contour can be computed as the area of the ellipse and it is given by,

$$\frac{\pi}{ab} = \frac{\pi}{\sqrt{1-\rho^2}} \quad (18)$$

The ratio of the area in the shaded region to the total area of the contour is given by,

$$\frac{1}{2\pi} cos^{-1}(\rho) \quad (19)$$

Observe that the ratio is independent of k, the constant for the contour. Hence, the probability of sign change $P(t^{-\to +})$ is given by the following expression.

$$P(t^{-\to +}) = \frac{1}{2\pi} cos^{-1}(\rho) \quad (20)$$

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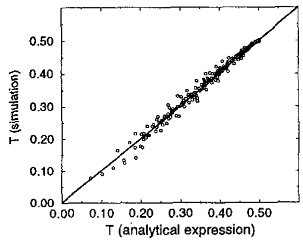

Figure 5: Validation of analytical expression for average bit transition activity

Theorem 2 relates the word-statistics to the probability of a sign change between consecutive word-values. The MSB bit is the sign-bit in one's-complement, two's-complement or the sign magnitude encoding of the signal. Hence, the MSB bit transition activity (t_{msb}) in the one's-complement, two's-complement or the sign magnitude encoding of the signal is given by,

$$t_{msb} = P(t^{-\to+}) + P(t^{+\to-}) = \frac{1}{\pi}cos^{-1}(\rho) \tag{21}$$

The dual bit-type (DBT) model for word-level signals was proposed in [2]. The DBT model has two break points, BP_0 and BP_1, that are computed using the word-level statistics of the signal. The expressions for the break points BP_0, BP_1 presented in [2] are given by Eqns. (22) and (23) below,

$$BP_0 = log_2(\sigma) + log_2(\sqrt{1-\rho^2} + |\rho|/8) \tag{22}$$

$$BP_1 = log_2(3\sigma) \tag{23}$$

In this work, we use the above break-points and Eqn. (21) to compute the average bit transition activity. The uniform white noise model is valid for the least significant bits up to the break point BP_0. The transition activity of these bits is 0.5. The transition activity of the sign bits, bits from the most significant bit to BP_1 is given by (21). For the bits between BP_0 and BP_1 a linear approximation between 0.5 and t_{msb} is used to obtain the bit transition activity. The average bit transition activity (T) is the sum of the bit transition activity values divided by the word-length. It is given by the following expression,

$$T = 0.5\ r\ +\ (1-r)\ \frac{1}{\pi}cos^{-1}(\rho) \tag{24}$$

r is the ratio of the average value of the break points BP_0, BP_1 and the signal word-length B. To experimentally validate the analytical expression for average bit transition activity, a first order auto-regressive model [3] was used to generate the signal word values for the given word-statistics. Fig. 5 shows a scatter plot of the total bit transition activity T obtained using the analytical expression and simulation for different values of word-length B (4, 8, 12, 16, 20, 24, 28, 32), ρ (0.0, 0.25, 0.5, 0.75, 0.9, 0.95, 0.99) and different σ^2 values. It can be seen that there is a good match between the results obtained by simulations and the analytical expression. The absolute average error was found to be 2.86%.

In the next section, we derive a relationship between the average bit transition activity and average bit temporal correlation.

4 Average Bit Temporal Correlation

In [3], an exact relation between the transition activity, probability and temporal correlation for a single bit signal was presented. This relation is given by (25):

$$t_i = 2p_i(1-p_i)(1-\rho_i), \tag{25}$$

where t_i is the transition activity, p_i is the probability and ρ_i is the temporal correlation of bit i. By corollary 1, the bit probability (p_i) of a zero-mean Gaussian signal with a two's complement encoding is 0.5 for all the bits. Substituting the values of p_i's in (25),

$$t_i = 0.5(1-\rho_i) \tag{26}$$

The average bit transition activity T is given by,

$$T = \frac{1}{B}\sum_{i=1}^{B} t_i \tag{27}$$

Substituting (26) in (27) we obtain,

$$T = 0.5 - \frac{0.5}{B}\sum_{i=1}^{B}\rho_i = 0.5 - 0.5\rho_{avg}. \tag{28}$$

Rearranging (28) we obtain the following relationship between the average bit transition activity (T) and the average bit temporal correlation (ρ_{avg}).

$$\rho_{avg} = 1 - 2T \tag{29}$$

Thus, ρ_{avg} can be computed from the word-statistics using (24) and (29).

5 Conclusion

In this paper, we presented analytical expressions for the bit statistics. We derived the result for the bit probability and the average bit probability. This was used to relate the average bit transition activity to the average bit temporal correlation. We developed a simple analytical expression for the average bit transition activity. We compared the average bit transition activity obtained by our model with a long simulation to verify the accuracy of the analytical expression. Hence, given the word-statistics of a signal, the bit statistics *average bit probability*, *average bit temporal correlation*, and *average bit transition activity* can be computed analytically.

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