On the Stability of Polygons of Polynomials with an Application : An Alternative Proof of the Edge Theorem

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Abstract

In this paper, we present some results related to the stability of polygons of real polynomials. These results are then used to test the stability of the two-dimensional exposed faces of a polytope of polynomials. The results are also used to give an alternative proof of the Edge Theorem.

1. Introduction

Kharitonov's Theorem [8] followed by the Edge Theorem of Bartlett, Hollot and Lin [3], have inspired a large number of research papers in the field of robust control, [2,4,9,10] being a small sample of this activity, and it is continuing to flourish due to its applications. All the existing techniques of checking the stability, of a polytope of polynomials [6,7,11], concentrate on the stability of the edges of the polytope. We have attacked this problem by checking the stability of the two dimensional faces of a polytope. First, we derive some necessary and sufficient conditions for a polynomial to vanish at $j\omega_0$, for some ω_0, in a polygon of polynomials, in terms of the winding number function (to be defined later) of the polygon. These results are then used to develop a method for checking the stability of polytope of polynomials, whose efficiency is comparable to that of Anagnost, Desoer and Minnichelli [1]. Using these techniques, an alternative proof of the Edge Theorem [3] is given.

2. Notations and Main Results

The formal definition of a standard polygon is now given.

Definition 2.1: A polygon of real polynomials will be called standard if

(i) it is convex, (ii) all the polynomials are of the same degree and (iii) its vertices are stable.

Notation 2.1 : Let $f^e(s) = f_e(s)$ and $f^o(s) = f_o(s)/s$ where $f_e(s)$ and $f_o(s)$ denote the usual even and odd parts of $f(s)$ respectively.

We now state a result of Chapellat and Bhattacharyya[5], which would be extensively used throughout the paper.

Theorem 2.1 [5]: *A convex combination of two polynomials , $f_1(s)$ and $f_2(s)$, vanishes at some $s = j\omega_0$ iff*

$$f_1^e(\omega_0)f_2^o(\omega_0) - f_1^o(\omega_0)f_2^e(\omega_0) = 0$$
$$f_1^e(\omega_0)f_2^e(\omega_0) \le 0$$
$$\text{and} \quad f_1^o(\omega_0)f_2^o(\omega_0) \le 0$$

Notation 2.2 : Suppose $A_1, A_2, A_3, \ldots, A_n$ are the vertices of a standard n-sided polygon and $f_1(s), f_2(s), \ldots, f_n(s)$ are the polynomials representing them. With each pair of vertices A_i, A_j of this polygon, we associate a frequency-dependent function $X_{i,j}(\omega)$ defined as follows:

$$X_{i,j}(\omega) = f_i^e(\omega)f_j^o(\omega) - f_j^e(\omega)f_i^o(\omega)$$

where $(i,j) \in S_n$, $S_n = \{(i,j) : i \in N_n, j \in N_n, i \neq j\}$, $N_n = \{i : 1 \le i \le n\}$. Note that $X_{i,j}(\omega)$ and $X_{j,i}(\omega)$ differ only by sign. Henceforth an n-sided polygon will be denoted by P_n. In order to specify the vertices of a triangle explictly, we employ the notation $\Delta_{i,j,k}$, where A_i, A_j, A_k are its vertices.

For the sake of convenience, we will employ the notation

$$\sum_{(i,j)=(1,2)}^{(n,1)} X_{i,j}(\omega) = X_{1,2}(\omega) + X_{2,3}(\omega) + \cdots + X_{n,1}(\omega)$$

Definition 2.2: For a standard n-sided polygon P_n, of polynomials, we define a frequency-dependent *winding number function* $S(\omega)$ as follows:-

$$S(\omega) = \sum_{(i,j)=(1,2)}^{(n,1)} Sgn(X_{i,j}(\omega))$$

where

$$Sgn(X_{i,j}(\omega)) = \begin{cases} 1 & \text{if } X_{i,j}(\omega) > 0 \\ -1 & \text{if } X_{i,j}(\omega) < 0 \\ 0 & \text{if } X_{i,j}(\omega) = 0 \end{cases}$$

Remark 2.1: It is easy to check that the sign of each $X_{i,j}(\omega)$ depends on the segment joining the end-points and not on the end-points.

We now state some theorems, which deal with a standard triangle of polynomials.

Proposition 2.1 : *In a standard polygon P_n, if $n-1$ sides have their corresponding $X_{i,j}(\omega_0)$'s equal to zero, for some ω_0, then the n^{th} side also has its $X_{i,j}(\omega_0)$ equal to zero.*

Theorem 2.2: *If a polynomial $f(s)$ in a standard triangle $\Delta_{1,2,3}$ vanishes at $j\omega_0$, for some ω_0, then*

$$\left(X_{i,j}(\omega_0) \ge 0, \forall \quad (i,j) \in S_3 \quad or \quad X_{i,j}(\omega_0) \le 0, \forall \quad (i,j) \in S_3\right)$$

Theorem 2.3: *If, for some ω_0*

$$\left(X_{i,j}(\omega_0) \ge 0, \forall \quad (i,j) \in S_3 \quad or \quad X_{i,j}(\omega_0) \le 0, \forall \quad (i,j) \in S_3\right)$$
$$and \quad \sum_{(i,j)=(1,2)}^{(3,1)} X_{i,j}(\omega_0) \neq 0$$

then there exists a unique polynomial $f(s)$ in $\Delta_{1,2,3}$, which vanishes at $j\omega_0$.

Now we present some corollaries derived from Theorem 2.2 and Theorem 2.3.

Cor. 2.1 : *If a polynomial within a standard triangle, P_3 vanishes at $j\omega_0$, then $S(\omega_0) = \pm 3$ or 0.*

Cor. 2.2 : *If $S(\omega_0) = \pm 3$, then there exists a unique polynomial, in the interior of the triangle, which vanishes at $j\omega_0$.*

Cor. 2.3 : *If a polynomial on an edge of a standard triangle, P_3, vanishes at $j\omega_0$ then $S(\omega_0) = \pm 2$ or 0.*

Cor. 2.4 : *If $S(\omega_0) = \pm 2$, then a polynomial on one edge of the standard triangle vanishes at $j\omega_0$.*

3. Results for a Polygon of Polynomials

Proposition 3.1 : *If $S(\omega_0) = 0$, for some ω_0, for a polygon P_n, then either there is no polynomial vanishing at $j\omega_0$ in P_n, or a line of polynomials, vanishing at $j\omega_0$, intersects the polygon.*

Definition 3.1 : A peripheral chord, $PC_{i,j}$, of P_n, is the line segment connecting the vertices A_i and A_j, of P_n, such that $|i-j| = 2$.

Theorem 3.1 : *If P_n is a standard polygon, and if $S(\omega_0)$ equals $\pm n$ or $\pm(n-1)$, for some ω_0, then there exists a unique polynomial $f(s)$, in P_n, which vanishes at $j\omega_0$.*

Proof :

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Note that Theorem 2.3 is a special case of Theorem 3.1, for $n = 3$. Therefore, we give the proof for $n \geq 4$.

We first prove that if $S(\omega_0) = \pm n$, then a polynomial vanishes in the interior of P_n.

Case I :*Assume without loss of generality that* $S(\omega_0) = \pm n$.

The proof will be given by induction for $n \geq 4$. Assume that the statement of Theorem 3.1 is true for $n-1$.

Partition the polygon P_n, by the peripheral chord $PC_{i-1,i+1}$, into a standard triangle, P_3 with A_{i-1}, A_i, A_{i+1} as its vertices and a standard polygon P_{n-1}. Without loss of generality assume that $S(\omega_0) = n$ for P_n.

We split the proof into two cases depending on the sign of $X_{i-1,i+1}(\omega_0)$.

Case I(a) : *Suppose* $X_{i-1,i+1}(\omega_0) = 0$.

By applying Cor. 2.4, to P_3, we conclude that the chord $PC_{i-1,i+1}$ has a polynomial $f(s)$ vanishing at $j\omega_0$. Since, this polynomial also lies within P_n, we are done.

Case I(b) : *Suppose* $X_{i-1,i+1}(\omega_0) \neq 0$.

If $X_{i-1,i+1}(\omega_0) > 0$, then $S(\omega_0) = n-1$, for P_{n-1}. Therefore, by the induction hypothesis, a polynomial within P_{n-1}, and hence within P_n, vanishes at $j\omega_0$. On the other hand, if $X_{i-1,i+1}(\omega_0) < 0$ then $S(\omega_0) = 3$ for P_3. By Cor. 2.2, a polynomial within P_3, and hence within P_n, vanishes at $j\omega_0$.

Case II : *Suppose* $S(\omega_0) = \pm(n-1)$.

Let A_iA_{i+1} be the side, which has $X_{i,i+1}(\omega_0) = 0$, while all other $X_{i,j}(\omega_0)$'s are non-zero and of the same sign. Without loss of generality, assume that $S(\omega_0) = n-1$ for P_n. Now, construct the peripheral chord $PC_{i-1,i+1}$, which partitions the polygon P_n, into a standard triangle, P_3 with A_{i-1}, A_i, A_{i+1} as its vertices and a standard polygon P_{n-1}.

It is clear that $X_{i-1,i+1}(\omega_0) \neq 0$, otherwise, by Prop. 2.1, P_3, which has two sides (A_iA_{i+1} and $A_{i-1}A_{i+1}$) with $X_{i,j}(\omega_0)$'s equal to zero, would have $X_{i-1,i}(\omega_0)$ equal to zero, for the third side. This contradicts the fact that for P_n, $S(\omega_0) = n-1$. Also, $X_{i-1,i+1}(\omega_0)$ cannot be positive, otherwise by Case I, we would have another polynomial vanishing at $j\omega_0$, in P_{n-1}, which would imply that at least two sides of P_n have their corresponding $X_{i,j}(\omega_0)$'s equal to zero. This would imply that $S(\omega_0) < n-1$ for P_n, which is a contradiction to the hypothesis. The only possiblilty is that $X_{i-1,i+1}(\omega_0) < 0$. Then, $S(\omega_0) = 2$, for P_3, and hence by Cor. 2.4 there exists a polynomial on the side A_iA_{i+1}, which vanishes at $j\omega_0$. Q.E.D.

We now, present some corollaries, obtained from Theorem 3.1, the proofs of which are easily derivable from the proof of Theorem 3.1.

Cor. 3.1 : *A unique polynomial vanishes in the interior of a standard polygon* P_n, *if* $S(\omega_0) = \pm n$ *for* P_n.

Cor. 3.2 : *A unique polynomial vanishes on an edge of a standard polygon* P_n, *if* $S(\omega_0) = \pm(n-1)$ *for* P_n.

Theorem 3.2 : *If a polynomial* $f(s)$ *in a standard polygon* P_n, *vanishes at* $j\omega_0$, *for some* ω_0, *then* $S(\omega_0)$ *equals* $\pm n, \pm(n-1)$ *or* 0.

4. Applications

First, we give an algorithm for checking the stability of a polygon of polynomials, which could be used to test the stability of a polytope of polynomials.

Algorithm for checking the stability of a polygon of polynomials.

Suppose the polygon has n sides.

(1) Compute $X_{i,j}(\omega)$, for each edge of the polygon. Find the zeros of each $X_{i,j}(\omega)$ and sketch their graphs against ω. As we are interested only in the signs of the $X_{i,j}(\omega)$'s, these graph don't have to be drawn accurately.

(2) If $S(\omega) \neq \pm n, \quad \pm(n-1)$ or 0, for any ω, then the polygon is stable.

(3) If $S(\omega_0) = 0$ for some ω_0, then either the polygon is stable or a line of polynomials intersects two edges of the polygon. This can be ascertained by applying Proposition 3.1.

Algorithm for checking the stability of a polytope of polynomials.

It is being assumed that the polytope's two-dimensional faces are explicitly known.

Check the stability of each two-dimensional face of the polytope by using the procedure as outlined above. If each two dimensional face is stable, then by the Edge Theorem [3], the polytope is stable.

As another application, we provide an alternative proof of the Edge Theorem [3].

Theorem 4.1 [3]: *If the exposed edges of a polytope of polynomials are stable, then the polytope is stable.*

We first need a technical lemma. Its proof will appear elsewhere.

Lemma 4.1 : *If for a standard polygon,* $S(\omega_0) = \pm n$ *for some* ω_0, *then there exists a frequency* ω_1 *such that at least one edge of the polygon has a polynomial vanishing at* $j\omega_1$.

Proof of the Edge Theorem [3] :

Suppose the claim is false and there exists a polynomial $g(s)$, within the polytope, which is unstable. By [Lemma 1,[3]], we can assume that there exists a polynomial $f(s)$, within a two-dimensional exposed face F of the polytope, which vanishes at $j\omega_0$ for some ω_0. Clearly, no other polynomial, $f_1(s)$ in F, can vanish at $j\omega_0$ because then the whole line joining $f_1(s)$ and $f(s)$, will vanish at $j\omega_0$ and intersect the stable edges of F. This would lead to a contradiction. Then by Theorem 3.2, either $S(\omega_0) = \pm n, \pm(n-1)$ or 0. If $S(\omega_0) = \pm(n-1)$, then by Cor. 3.2, there is a polynomial on an edge of the polygon vanishing at $j\omega_0$, which is a contradiction. If $S(\omega_0) = 0$, then either there is no polynomial in F, vanishing at $j\omega_0$ or a line of polynomials vanishing at $j\omega_0$, which intersects the face F. Either of these two possibilities leads to a contradiction. Consider now the only remaining case of $S(\omega_0) = \pm n$. In this case, by Lemma 4.1, there exists a frequency ω_1 such that at least one of the edges of the polygon has a polynomial vanishing at $j\omega_1$. But this also is not possible as the edges are stable. Q.E.D.

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